Price of Stability in Polynomial Congestion Games*

George Christodoulou   Martin Gairing

December 10, 2013

Abstract

The Price of Anarchy in congestion games has attracted a lot of research over the last decade. This resulted in a thorough understanding of this concept. In contrast the Price of Stability, which is an equally interesting concept, is much less understood.

In this paper, we consider congestion games with polynomial cost functions with nonnegative coefficients and maximum degree d. We give matching bounds for the Price of Stability in such games, i.e., our technique provides the exact value for any degree d.

For linear congestion games, tight bounds were previously known. Those bounds hold even for the more restricted case of dominant equilibria, which may not exist. We give a separation result showing that already for congestion games with quadratic cost functions this is not possible; that is, the Price of Anarchy for the subclass of games that admit a dominant strategy equilibrium is strictly smaller than the Price of Stability for the general class.

1 Introduction

During the last decade, the quantification of the inefficiency of game-theoretic equilibria has been a popular and successful line of research. The two most widely adopted measures for this inefficiency are the Price of Anarchy (PoA) [17] and the Price of Stability (PoS) [3].

Both concepts compare the social cost in a Nash equilibrium to the optimum social cost that could be achieved via central control. The PoA is pessimistic and considers the worst-case such Nash equilibrium, while the PoS is optimistic and considers the best-case Nash equilibrium. Therefore, the PoA can be used as an absolute worst-case guarantee in a scenario where

*This work was supported by EPSRC grants EP/K01000X/1 and EP/J019399/1.

we have no control over equilibrium selection. On the other hand, the PoS gives an estimate of what is the best we can hope for in a Nash equilibrium; for example, if the players collaborate to find the optimal Nash equilibrium, or if a trusted mediator suggest this solution to them. Moreover, it is a much more accurate measure for those instances that possess unique Nash equilibria.

Congestion games [21] have been a driving force in recent research on these inefficiency concepts. In a congestion game, we are given a set of resources and each player selects a subset of them (e.g. a path in a network). Each resource has a cost function that only depends on the number of players that use it. Each player aspires to minimise the sum of the resources' costs in its strategy given the strategies chosen by the other players. Congestion games always admit a *pure* Nash equilibrium [21], where players pick a single strategy and do not randomize. Rosenthal [21] showed this by means of a *potential function* having the following property: if a single player deviates to a different strategy then the value of the potential changes by the same amount as the cost of the deviating player. Pure Nash equilibria correspond to local optima of the potential function. Games admitting such a potential function are called potential games and every potential game is isomorphic to a congestion game [20].

Today we have a strong theory which provides a thorough understanding of the PoA in congestion games [1, 4, 5, 11, 22]. This theory includes the knowledge of the exact value of the PoA for games with linear [4, 11] and polynomial [1] cost functions, a recipe for computing the PoA for general classes of cost functions [22], and an understanding of the "complexity" of the strategy space required to achieve the worst case PoA [5].

In contrast, we still only have a very limited understanding of the Price of Stability (PoS) in congestion games. Exact values for the PoS are only known for congestion games with linear cost functions [9, 14] and certain network cost sharing games [3]. The reason for this is that there are more considerations when bounding the PoS as compared to bounding the PoA. For example, for linear congestion games, the techniques used to bound the PoS are considerably more involved than those used to bound the PoA.

A fundamental concept in the design of games is the notion of a *dominant*-strategy equilibrium. In such an equilibrium each player chooses a strategy which is better than any other strategy no matter what the other players do. It is well-known that such equilibria do not always exist, as the requirements imposed are too strong. However, it is appealing for a game designer, as it makes outcome prediction easy. It also simplifies the strategic reasoning of the players and is therefore an important concept in mechanism design. If we restrict to instances where such equilibria exist, it is natural to ask how inefficient those equilibria can be. Interestingly, for linear congestion games, they can be as inefficient as the PoS [9, 14, 12].

1.1 Contribution and High-Level Idea

Results. In this paper we study the fundamental class of congestion games with polynomial cost functions of maximum degree d and nonnegative coefficients. Our main result reduces the problem of finding the value of the Price of Stability to a single-parameter optimization problem. It can be summarized in the following theorem (which combines Theorem 3.3 and Theorem 4.5).

Theorem 1.1 *For congestion games with polynomial cost functions with maximum degree d and nonnegative coefficients, the Price of Stability is given by*

$$\mathsf{PoS} = \max_{r>1} \frac{(2^d d + 2^d - 1) \cdot r^{d+1} - (d+1) \cdot r^d + 1}{(2^d + d - 1) \cdot r^{d+1} - (d+1) \cdot r^d + 2^d d - d + 1}.$$

For any degree d, this gives the exact value of the Price of Stability. For example, for $d = 1$ and $d = 2$, we get

$$\max_r \frac{3\,r^2 - 2\,r + 1}{2\,r^2 - 2\,r + 2} = 1 + \frac{\sqrt{3}}{3} \approx 1.577 \quad \text{and} \quad \max_r \frac{11\,r^3 - 3\,r^2 + 1}{5\,r^3 - 3\,r^2 + 7} \approx 2.36,$$

respectively. The PoS converges to $d + 1$ for large d.

We further show that in contrast to linear congestion games [14, 12], already for $d = 2$, there is no instance which admits a dominant strategy equilibrium and achieves this value. More precisely, we show in Theorem 5.2 that for the subclass of games that admit a dominant strategy equilibrium the Price of Anarchy is strictly smaller than the Price of Stability for the general class.

Upper Bound Techniques. Both finding upper and lower bounds for the PoS, seem to be a much more complicated task than bounding the PoA. For the PoA of a class of games, one needs to capture the worst-case example of *any* Nash equilibrium, and the PoA methodology has been heavily based on this fact. On the other hand, for the PoS of the same class one needs to capture the worst-case instance of the *best* Nash equilibrium. So far, we do not know a useful characterization of the set of best-case Nash equilibria.

It is not straightforward to transfer the techniques for the PoA to solve the respective PoS problem.

A standard approach that has been followed for upper bounding the PoS can be summarised as follows:

1. Define a restricted subset $\mathcal{R}$ of Nash equilibria.
2. Find the Price of Anarchy with respect to Nash equilibria that belong in $\mathcal{R}$.

The above recipe introduces new challenges: What is a good choice for $\mathcal{R}$, and more importantly, how can we incorporate the description of $\mathcal{R}$ in the Price of Anarchy methodology? For example, if $\mathcal{R}$ is chosen to be the set of *all* Nash equilibria, then one obtains the PoA bound. Finding an appropriate restriction is a non-trivial task and might depend on the nature of the game, so attempts vary in the description level of $\mathcal{R}$ from natural, "as the set of equilibria with optimum potential", to the rather more technical definitions like "the equilibria that can be reached from a best-response path starting from an optimal setup".

Like previous work (see for example [3, 6, 9, 14, 12]) we consider the PoA of Nash equilibria with minimum potential (or in fact with potential smaller than the one achieved in the optimum).

Then we use a linear combination of two inequalities, which are derived from the potential and the Nash equilibrium conditions, respectively. Using only the Nash inequality gives the PoA value [1]. Using only the potential inequality gives an upper bound of $d+1$. The question is what is the best way to combine these inequalities to obtain the minimum possible upper bound? Caragiannis et al. [9] showed how to do this for linear congestion games. Our analysis shows how to combine them optimally for all polynomials (cf. parameter $\widehat{\nu}$ in Definition 4.2).

The main technical challenge is to extend the techniques used for proving upper bounds for the PoA [1, 11, 22]. In general those techniques involve optimizing over two parameters λ, μ such that the resulting upper bound on the PoA is minimized and certain technical conditions are satisfied – Roughgarden [22] refers to those conditions as (λ, μ)-smoothness. The linear combination of the two inequalities mentioned above adds a third parameter ν, which makes the analysis much more involved.

Lower Bound Techniques. Proving lower bounds for the PoA and PoS is usually done by constructing specific classes of instances. However, there is a conceptual difference: Every Nash equilibrium provides a lower bound on the PoA, while for the PoS we need to give a Nash equilibrium and prove that this is the best Nash equilibrium. To guarantee optimality, the main approach is based on constructing games with *unique* equilibria. One way to guarantee this is to define a game with a dominant-strategy equilibrium. This approach gives tight lower bounds in congestion games with linear cost functions [14, 12]. Recall, that our separation result (Theorem 5.2) shows that, already for $d = 2$, dominant-strategy equilibria will not give us a tight lower bound. Thus, we use a different approach. We construct an instance with a unique Nash equilibrium and show this by using an inductive argument (Lemma 3.2).

The construction of our lower bound was governed by the inequalities used in the proof of the upper bound. At an abstract level, we have to construct an instance that uses the cost functions and loads on the resource that make all used inequalities tight. This is not an easy task as there are many inequalities: most prominently, one derived from the Nash equilibrium condition, one from the potential, and a third one that upper bounds a linear combination of them (see Proposition 4.4). To achieve this we had to come up with a completely novel construction.

1.2 Related Work

The term Price of Stability[1] was introduced by Anshelevich et al. [3] for a network design game, which is a congestion game with special decreasing cost functions. For such games with n players, they showed that the Price of Stability is exactly H_n, i.e., the n'th harmonic number. For the special case of *undirected* networks, the PoS is known to be strictly smaller than H_n [15, 7, 13, 3], but while the best general upper bound [15] is close to H_n, the best current lower bound is a constant [8]. For special cases better upper bound can be achieved. Li [19] showed an upper bound of $O(\log n / \log \log n)$ when the players share a common sink, while Fiat et al. [16] showed a better upper bound of $O(\log \log n)$ when in addition there is a player in every vertex of the network. This was very recently improved to $O(\log \log n)$ by [18]. Chen and Roughgarden [10] studied the PoS for the *weighted* variant of this game, where each player pays for a share of each edge cost proportional to her weight, and Albers [2] showed that the PoS is $\Omega(\log W / \log \log W)$, where W is the sum of the players' weights.

The PoS has also been studied in congestion games with increasing cost functions. For linear congestion games, the PoS is equal to $1 + \sqrt{3}/3 \approx$

[1]Notice that this concept has been studied already before, see e.g. [23].

1.577 where the lower bound was shown in [14] and the upper bound in [9]. Bilo[6] showed upper bounds on the PoS of 2.362 and 3.322 for congestion games with quadratic and cubic functions respectively. He also gives non-matching lower bounds, which are derived from the lower bound for linear cost functions in [12].

Awerbuch et al. [4] and Christodoulou and Koutsoupias [11] showed that the PoA of congestion games with linear cost functions is $\frac{5}{2}$. Aland et. al. [1] obtained the exact value on the PoA for polynomial cost functions. Roughgarden's [22] smoothness framework determines the PoA with respect to any set of allowable cost functions. These results have been extended to the more general class of *weighted* congestion games [1, 4, 5, 11].

Roadmap. The rest of the paper is organized as follows. In Section 2 we introduce polynomial congestion games. In Section 3 and 4, we present our matching lower and upper bounds on the PoS. We present a separation result in Section 5. Due to space constraints, some of the proofs are deferred to the full version of this paper.

2 Definitions

For any positive integer $k \in \mathbb{N}$, denote $[k] = \{1, \ldots, k\}$. A *congestion game* [21] is a tuple $(N, E, (\mathcal{S}_i)_{i \in N}, (c_e)_{e \in E})$, where $N = [n]$ is a set of n players and E is a set of resources. Each player chooses as her pure *strategy* a set $s_i \subseteq E$ from a given *set of available strategies* $S_i \subseteq 2^E$. Associated with each resource $e \in E$ is a nonnegative *cost function* $c_e : \mathbb{N} \to \mathbb{R}^+$. In this paper we consider polynomial cost functions with maximum degree d and nonnegative coefficients; that is every cost function is of the form $c_e(x) = \sum_{j=0}^{d} a_{e,j} \cdot x^j$ with $a_{e,j} \geq 0$ for all j.

A *pure strategy profile* is a choice of strategies $\mathsf{s} = (s_1, s_2, ...s_n) \in \mathcal{S} = \mathcal{S}_1 \times \cdots \times \mathcal{S}_n$ by players. We use the standard notation $\mathsf{s}_{-i} = (s_1, \ldots, s_{i-1}, s_{i+1}, \ldots s_n)$, $\mathcal{S}_{-i} = \mathcal{S}_1 \times \cdots \times S_{i-1} \times S_{i+1} \times \cdots \times \mathcal{S}_n$, and $\mathsf{s} = (s_i, \mathsf{s}_{-i})$. For a pure strategy profile s define the *load* $n_e(\mathsf{s}) = |i \in N : e \in s_i|$ as the number of players that use resource e. The *cost* for player i is defined by $C_i(\mathsf{s}) = \sum_{e \in s_i} c_e(n_e(\mathsf{s}))$.

Definition 2.1 A pure strategy profile s is a pure *Nash equilibrium* if and only if for every player $i \in N$ and for all $s'_i \in \mathcal{S}_i$, we have $C_i(\mathsf{s}) \leq C_i(s'_i, \mathsf{s}_{-i})$.

Definition 2.2 A pure strategy profile s is a (weakly) *dominant strategy equilibrium* if and only if for every player $i \in N$ and for all $s'_i \in \mathcal{S}_i$ and $\mathsf{s}_{-i} \in \mathcal{S}_{-i}$, we have $C_i(\mathsf{s}) \leq C_i(s'_i, \mathsf{s}_{-i})$.

The *social cost* of a pure strategy profile s is the sum of the players costs

$$\mathrm{SC}(\mathsf{s}) = \sum_{i \in N} C_i(\mathsf{s}) = \sum_{e \in E} n_e(\mathsf{s}) \cdot c_e(n_e(\mathsf{s})).$$

Denote $\text{OPT} = \min_{\mathsf{s}} SC(\mathsf{s})$ as the *optimum social cost* over all strategy profiles $\mathsf{s} \in \mathcal{S}$. The *Price of Stability* of a congestion game is the social cost of the best-case Nash equilibrium over the optimum social cost

$$\mathsf{PoS} = \min_{\mathsf{s} \text{ is a Nash Equilibrium}} \frac{\mathrm{SC}(\mathsf{s})}{\mathrm{OPT}}.$$

The PoS for a class of games is the largest PoS among all games in the class.

For a class of games that admit dominant strategy equilibria, the *Price of Anarchy of dominant strategies*, dPoA, is the worst case ratio (over all games) between the social cost of the dominant strategies equilibrium and the optimum social cost.

Congestion games admit a potential function $\Phi(\mathsf{s}) = \sum_{e \in E} \sum_{j=1}^{n_e(\mathsf{s})} c_e(j)$ which was introduced by Rosenthal [21] and has the following property: for any two strategy profiles s and (s'_i, s_{-i}) that differ only in the strategy of player $i \in N$, we have $\Phi(\mathsf{s}) - \Phi(s'_i, \mathsf{s}_{-i}) = C_i(\mathsf{s}) - C_i(s'_i, \mathsf{s}_{-i})$. Thus, the set of pure Nash equilibria correspond to local optima of the potential function. More importantly, there exists a pure Nash eqilibrium s, s.t.

$$\Phi(\mathsf{s}) \leq \Phi(\mathsf{s}') \quad \text{for all } \mathsf{s}' \in S. \tag{1}$$

3 Lower bound

In this section we use the following instance to show a lower bound on PoS.

Example 3.1 Given nonnegative integers n, k and d, define a congestion game as follows:

- The set of resources E is partitioned into $E = \mathcal{A} \cup \mathcal{B} \cup \{\Gamma\}$ where $\mathcal{A}$ consists of n resources $\mathcal{A} = \{A_i | i \in [n]\}$, $\mathcal{B}$ consists of $n(n-1)$ resources $\mathcal{B} = \{B_{ij} | i, j \in [n], i \neq j\}$, and Γ is a single resource.
- All cost functions are monomials of degree d given as follows:
  - For $i \in [n]$ the cost of resource A_i is given by $c_{A_i}(x) = \alpha_i \cdot x^d$, where

$$\alpha_i \quad = \quad (k+i)^d + \varepsilon \qquad \text{for sufficiently small } \varepsilon > 0.$$

– Denote $T_i = \frac{(k+i)^d-(k+i-1)^d}{(2^{2d}-1)}$. Resource B_{ij} with $i,j \in [n], i \neq j$ has cost

$$c_{B_{ij}}(x) = \beta_{ij} \cdot x^d \qquad \text{where } \beta_{ij} = \begin{cases} T_j & \text{, if } i < j, \\ 2^d T_i & \text{, if } i > j. \end{cases}$$

– For resource Γ we have $c_\Gamma(x) = x^d$.

- There are $n+k$ players. Each player $i \in [n]$ has two strategies s_i, s_i^* where

$$s_i = \Gamma \cup \{B_{ij} | j \in [n], j \neq i\} \text{ , and}$$
$$s_i^* = A_i \cup \{B_{ji} | j \in [n], j \neq i\}.$$

  The remaining players $i \in [n+1, n+k]$ are fixed to choose the single resource Γ. To simplify notation denote by $\mathsf{s} = (s_1, \ldots, s_n)$ and $\mathsf{s}^* = (s_1^*, \ldots, s_n^*)$ the corresponding strategy profiles. Those profiles correspond to the unique Nash equilibrium and to the optimal allocation respectively.

In the following lemma we show that s is the unique Nash equilibrium for the game in Example 3.1. To do so, we show that s_1 is a dominant strategy for player 1 and that given that the first $i-1$ players play $s_1, \ldots, s_{i-1}$, then s_i is a dominant strategy for player $i \in [n]$.

Lemma 3.2 *In the congestion game from Example 3.1, s is the unique Nash equilibrium.*

Proof: We will show that s_1 is a dominant strategy for the first player and that given that the first $i-1$ players play $s_1, \ldots, s_{i-1}$, then s_i is a dominant strategy for player i, for $i = 1 \ldots n$.

Let $(s_i, \mathsf{s}^{i-1}, j)$ and $(s_i^*, \mathsf{s}^{i-1}, j)$ be the strategy profiles where the first $i-1$ players and j more players play their Nash Equilibrium strategies, while player i plays s_i and s_i^* respectively. Notice that $i \in [1, n-1]$ and $j \in [0, n-i]$. Let's denote the set of the j players by J.

$$C_i(s_i, \mathsf{s}^{i-1}, j) \;=\; (k+j+i)^d + \sum_{h \leq i-1} \beta_{ih} + \sum_{h \in J} \beta_{ih} + 2^d \cdot \sum_{h > i \text{ and } h \notin J} \beta_{ih}.$$

Similarly

$$C_i(s_i^*, \mathsf{s}^{i-1}, j) \;=\; \alpha_i + 2^d \sum_{h \leq i-1} \beta_{hi} + 2^d \sum_{h \in J} \beta_{hi} + \sum_{h > i \text{ and } h \notin J} \beta_{hi}.$$

We need to show that $C_i(s_i, \mathsf{s}^{i-1}, j) < C_i(s_i^*, \mathsf{s}^{i-1}, j)$ for all $i = 1, \ldots, n$ and $j = 0, \ldots, n-i$. Equivalently it is enough to show that

$$\begin{aligned}
\alpha_i > A_{ij} :=& (k+i+j)^d + \sum_{h \le i-1} \Big(\beta_{ih} - 2^d \beta_{hi}\Big) + \sum_{h \in J} \Big(\beta_{ih} - 2^d \beta_{hi}\Big) \\
& + \sum_{h > i \text{ and } h \notin J} \Big(2^d \beta_{ih} - \beta_{hi})\Big) \\
=& (k+i+j)^d + \sum_{h \in J} \Big(\beta_{ih} - 2^d \beta_{hi}\Big) \\
=& (k+i+j)^d + \Big(1 - 2^{2d}\Big) \sum_{h \in J} T_h \\
=& (k+i+j)^d - \sum_{h \in J} \Big((k+h)^d - (k+h-1)^d\Big).
\end{aligned}$$

By convexity of $(k+h)^d$, A_{ij} is maximized when $J = \{i+1, \ldots, i+j\}$, so

$$A_{ij} \quad \le \quad (k+i+j)^d - \sum_{i+1 \le h \le i+j} \Big((k+h)^d - (k+h-1)^d\Big) = (k+i)^d < \alpha_i.$$

The claim follows. ■

We use the instance from Example 3.1 to show the lower bound in the following theorem. We define $\rho = \frac{k}{n}$ and $r = \frac{k+n}{k} = 1 + \frac{1}{\rho} > 1$. We let $n \to \infty$ and determine the $r > 1$ which maximises the resulting lower bound[2]. Note that $r > 1$ is the ratio of the loads on resource Γ in Example 3.1.

Theorem 3.3 *For congestion games with polynomial cost functions with maximum degree d and nonnegative coefficients, we have*

$$\mathsf{PoS} \ge \max_{r > 1} \frac{(2^d d + 2^d - 1) \cdot r^{d+1} - (d+1) \cdot r^d + 1}{(2^d + d - 1) \cdot r^{d+1} - (d+1) \cdot r^d + 2^d d - d + 1}. \tag{2}$$

Proof: We use the instance from Example 3.1. To show the lower bound we define $\rho = \frac{k}{n}$ and $r = \frac{k+n}{k} = 1 + \frac{1}{\rho} > 1$, we let $n \to \infty$, and we determine the $r > 1$ which maximises the resulting lower bound. Note that $r > 1$ is the ratio of the loads on resource Γ in Example 3.1.

[2]Notice that the value r that optimizes the right hand side expression of (2) might not be rational. The lower bound is still valid as we can approximate an irrational r arbitrarily close by a rational.

For every subset $D \subseteq E$, denote $\mathrm{SC}_D(\mathsf{s}) = \sum_{e \in D} n_e(\mathsf{s}) \cdot c_e(n_e(\mathsf{s}))$ the contribution of resources in D to $\mathrm{SC}(\mathsf{s})$.

By Lemma 3.2, s is the unique Nash equilibrium. So, it suffices to lower bound $\frac{SC(\mathsf{s})}{SC(\mathsf{s}^*)}$. First observe that

$$\begin{aligned}\frac{SC(\mathsf{s})}{SC(\mathsf{s}^*)} &= \frac{SC_{\mathcal{B}}(\mathsf{s}) + SC_{\Gamma}(\mathsf{s})}{SC_{\mathcal{B}}(\mathsf{s}^*) + SC_{\Gamma}(\mathsf{s}^*) + SC_{\mathcal{A}}(\mathsf{s}^*)} \\ &= \frac{\frac{SC_{\mathcal{B}}(\mathsf{s})}{n^{d+1}} + \frac{SC_{\Gamma}(\mathsf{s})}{n^{d+1}}}{\frac{SC_{\mathcal{B}}(\mathsf{s}^*)}{n^{d+1}} + \frac{SC_{\Gamma}(\mathsf{s}^*)}{n^{d+1}} + \frac{SC_{\mathcal{A}}(\mathsf{s}^*)}{n^{d+1}}}\,. \end{aligned} \tag{3}$$

To determine the limit of $\frac{SC(\mathsf{s})}{SC(\mathsf{s}^*)}$ for $n \to \infty$, we can determine the limit of each term independently. Since $n_e(\mathsf{s}^*) = 1$ for all $e \in \mathcal{A}$,

$$\begin{aligned}\lim_{n\to\infty} \frac{SC_A(\mathsf{s}^*)}{n^{d+1}} &= \lim_{n\to\infty} \frac{\sum_{i\in[n]} c_{A_i}(1)}{n^{d+1}} = \lim_{n\to\infty} \frac{\sum_{i\in[n]} (k+i)^d}{n^{d+1}} \\ &= \lim_{n\to\infty} \frac{\sum_{i\in[n]} \left(\rho + \frac{i}{n}\right)^d}{n} = \int_{\rho}^{\rho+1} x^d dx = \frac{(1+\rho)^{d+1} - \rho^{d+1}}{d+1}. \end{aligned} \tag{4}$$

Each resource in $\mathcal{B}$ is used by exactly one player in s and also in s^*. So,

$$\begin{aligned}\lim_{n\to\infty} \frac{SC_{\mathcal{B}}(\mathsf{s})}{n^{d+1}} &= \lim_{n\to\infty} \frac{SC_{\mathcal{B}}(\mathsf{s}^*)}{n^{d+1}} \\ &= \lim_{n\to\infty} \frac{\sum_{h\le n} \sum_{i\le n, i\ne h} \beta_{ih}}{n^{d+1}} \\ &= \lim_{n\to\infty} \frac{\sum_{h\le n} (2^d+1)\cdot(h-1)\cdot T_h}{n^{d+1}} \\ &= \lim_{n\to\infty} (2^d+1)\cdot \sum_{h\le n} (h-1) \frac{\frac{(k+h)^d - (k+h-1)^d}{(2^{2d}-1)}}{n^{d+1}} \\ &= \frac{1}{(2^d-1)} \lim_{n\to\infty} \sum_{h\le n} (h-1) \frac{(k+h)^d - (k+h-1)^d}{n^{d+1}} \\ &= \frac{1}{(2^d-1)} \lim_{n\to\infty} \frac{n\,(k+n)^d - \sum_{h\le n} (k+h)^d}{n^{d+1}} \\ &= \frac{1}{(2^d-1)} \left((1+\rho)^d - \frac{(1+\rho)^{d+1} - \rho^{d+1}}{d+1} \right), \end{aligned} \tag{5}$$

where the last step is similar to (4). Moreover,

$$\lim_{n\to\infty}\frac{SC_\Gamma(\mathsf{s})}{n^{d+1}} = \lim_{n\to\infty}\frac{(n+k)^{d+1}}{n^{d+1}} = (1+\rho)^{d+1}, \tag{6}$$

and

$$\lim_{n\to\infty}\frac{SC_\Gamma(\mathsf{s}^*)}{n^{d+1}} = \lim_{n\to\infty}\frac{k^{d+1}}{n^{d+1}} = \rho^{d+1}. \tag{7}$$

Combining identities (3)-(7) yields

$$\begin{aligned}
&\lim_{n\to\infty}\frac{SC(\mathsf{s})}{SC(\mathsf{s}^*)} = \\
&= \frac{(1+\rho)^d - \frac{(1+\rho)^{d+1}-\rho^{d+1}}{d+1} + (2^d-1)\cdot(1+\rho)^{d+1}}{(1+\rho)^d - \frac{(1+\rho)^{d+1}-\rho^{d+1}}{d+1} + (2^d-1)\cdot\rho^{d+1} + (2^d-1)\cdot\frac{(1+\rho)^{d+1}-\rho^{d+1}}{d+1}} \\
&= \frac{(1+\rho)^{d+1}\big((d+1)(2^d-1)-1\big)+\rho^{d+1}+(d+1)(1+\rho)^d}{(1+\rho)^{d+1}(2^d-2)+\rho^{d+1}\left(d(2^d-1)+1\right)+(d+1)(1+\rho)^d}.
\end{aligned} \tag{8}$$

Recall that we defined $r = \frac{k+n}{k} = 1+\frac{1}{\rho} > 1$. Thus $\rho = \frac{1}{r-1}$ and $(1+\rho) = \frac{r}{r-1}$. Substituting this into (8) and multiplying by $(r-1)^{d+1}$ gives

$$\begin{aligned}
\lim_{n\to\infty}\frac{SC(\mathsf{s})}{SC(\mathsf{s}^*)} &= \frac{r^{d+1}\big((d+1)(2^d-1)-1\big)+1+(d+1)r^d(r-1)}{r^{d+1}(2^d-2)+\left(d(2^d-1)+1\right)+(d+1)r^d(r-1)} \\
&= \frac{(2^d d+2^d-1)r^{d+1}-(d+1)r^d+1}{(2^d+d-1)r^{d+1}-(d+1)r^d+2^d d-d+1},
\end{aligned}$$

which proves the lower bound. ■

4 Upper bound

In this section we show an upper bound on the PoS for polynomial congestion games. We start with two technical lemmas and a definition, all of which will be used in the proof of Proposition 4.4. This proposition is the most technical part of the paper. It shows an upper bound on a linear combination of two expressions; one is derived from the Nash equilibrium condition and the other one from the potential. Equipped with this, we prove our upper bound in Theorem 4.5.

Lemma 4.1 *Let f be a nonnegative and convex function, then for all nonnegative integers x, y with $x \geq y$, $\sum_{i=y+1}^{x} f(i) \geq \int_{y}^{x} f(t)dt + \frac{1}{2}(f(x) - f(y))$.*

Proof: The claim trivially holds for $x = y$. Since f is nonnegative and convex, for all j, we have

$$\int_{j}^{j+1} f(t)dt \leq \frac{1}{2}(f(j+1) + f(j)).$$

Summing up over all integer $j \in [y, x-1]$ proves the lemma. ■

Definition 4.2 Define $\widehat{\nu}$ as the minimum ν such that

$$f(\nu) := \left(2^d + (d-1)\left(1 - \frac{1}{r^{d+1}}\right) - \frac{1}{r}\right) \cdot \nu - d\left(1 - \frac{1}{r^{d+1}}\right) \geq 0$$

for all $r > 1$.

Observe that for all $d \geq 1$ and $r > 1$, $f(\nu)$ is a monotone increasing function in ν. Thus $\widehat{\nu} \in (0, 1]$ is well defined since $f(0) < 0$ and $f(1) > 0$ for all $r > 1$. Moreover, $f(\nu) \geq 0$ for all $\nu \geq \widehat{\nu}$. We will make use of the following bounds on $\widehat{\nu}$.

Lemma 4.3 *Define $\widehat{\nu}$ as in Definition 4.2. Then $\frac{d}{2^d+d-1} \leq \widehat{\nu} < \frac{d+1}{2^d+d-1}$.*

Proof: The lower bound follows directly from Definition 4.2 with $r \to \infty$. To see the upper bound recall that $f(\nu)$ is monotone increasing in ν, thus, it suffices to show that $f(\frac{d+1}{2^d+d-1}) > 0$ for all $r > 1$. Indeed

$$\begin{aligned} f(\frac{d+1}{2^d+d-1}) &= \left(2^d + (d-1)\left(1 - \frac{1}{r^{d+1}}\right) - \frac{1}{r}\right) \cdot \frac{d+1}{2^d+d-1} - d\left(1 - \frac{1}{r^{d+1}}\right) \\ &= d + 1 - \frac{(d-1)(d+1)}{r^{d+1}(2^d+d-1)} - \frac{d+1}{r(2^d+d-1)} - d + \frac{d}{r^{d+1}} \\ &> 0, \end{aligned}$$

since $r > 1$ and for all integer $d \geq 1$, we have $\frac{(d-1)(d+1)}{2^d+d-1} \leq d$ and $\frac{d+1}{2^d+d-1} \leq 1$. ■

Proposition 4.4 Let $1 \geq \nu \geq \widehat{\nu}$ and define $\lambda = d + 1 - d\nu$ and $\mu = (2^d + d - 1)\nu - d$. Then for all polynomial cost functions c with maximum degree d and nonnegative coefficients and for all nonnegative integers x, y we have

$$\nu \cdot y \cdot c(x+1) + (1-\nu)(d+1)\left(\sum_{i=1}^{y} c(i) - \sum_{i=1}^{x} c(i)\right) \leq (\mu + \nu - 1) \cdot x \cdot c(x) + \lambda \cdot y \cdot c(y).$$

Proof: Since c is a polynomial cost function with maximum degree d and nonnegative coefficients it is sufficient to show the claim for all monomials of degree t where $0 \leq t \leq d$. Thus, we will show that

$$(\mu+\nu-1)\cdot x^{t+1} + \lambda\cdot y^{t+1} - \nu\cdot y(x+1)^t + (1-\nu)(d+1)\left(\sum_{i=1}^{x} i^t - \sum_{i=1}^{y} i^t\right) \geq 0 \quad (9)$$

for all nonnegative integers x, y and degrees $0 \leq t \leq d$.

Fix some $0 \leq t \leq d$. First observe that (9) is trivially fulfilled for $y = 0$, as all the negative terms disappear. So in the following we assume $y \geq 1$.

Elementary calculations show that (9) holds when $0 \leq x \leq y$. For a proof see Lemma 6.1 in the appendix. So in the following we assume $x > y \geq 1$. By Lemma 4.1, we have

$$\begin{aligned}\sum_{i=1}^{x} i^t - \sum_{i=1}^{y} i^t = \sum_{i=y+1}^{x} i^t &\geq \frac{1}{t+1}(x^{t+1} - y^{t+1}) + \frac{1}{2}(x^t - y^t)\\ &\geq \frac{1}{d+1}(x^{t+1} - y^{t+1}) + \frac{1}{2}(x^t - y^t). \qquad (10)\end{aligned}$$

Moreover, since $x \geq 2$ we can bound

$$\begin{aligned}(x+1)^t = \sum_{i=0}^{t}\binom{t}{i}x^{t-i} &\leq x^t + x^{t-1}\cdot\sum_{i=1}^{t}\binom{t}{i}\left(\frac{1}{2}\right)^{i-1}\\ &= x^t + x^{t-1}\cdot 2\left(\left(\frac{3}{2}\right)^t - 1\right). \qquad (11)\end{aligned}$$

Using (10) and (11) and by defining $r = \frac{x}{y} > 1$, we can lower bound the left-hand-side of (9) by

$$\begin{aligned}&\mu\cdot x^{t+1} + (\lambda+\nu-1)\cdot y^{t+1} - \nu\cdot y(x+1)^t + \frac{1}{2}(1-\nu)(d+1)(x^t - y^t)\\ &\geq \left(\mu + \frac{\lambda+\nu-1}{r^{t+1}}\right)\cdot x^{t+1} - \frac{\nu}{r}\cdot\left(x^{t+1} + x^t\cdot 2\left(\left(\frac{3}{2}\right)^t - 1\right)\right)\\ &\quad + \frac{1}{2}(1-\nu)(d+1)\left(1 - \frac{1}{r^t}\right)\cdot x^t\\ &= \underbrace{\left(\mu + \frac{\lambda+\nu-1}{r^{t+1}} - \frac{\nu}{r}\right)}_{:=A(\nu)}\cdot x^{t+1} + \underbrace{\left(\frac{1}{2}(1-\nu)(d+1)\left(1-\frac{1}{r^t}\right) - \frac{2\nu}{r}\left(\left(\frac{3}{2}\right)^t - 1\right)\right)}_{:=B(\nu)}\cdot x^t.\end{aligned}$$

First observe that by using the definitions of λ, μ, we get

$$\begin{aligned}
A(\nu) &= \left(2^d + (d-1)\left(1 - \frac{1}{r^{t+1}}\right) - \frac{1}{r}\right) \cdot \nu - d\left(1 - \frac{1}{r^{t+1}}\right) \\
&\geq \left(2^d + (d-1)\left(1 - \frac{1}{r^{d+1}}\right) - \frac{1}{r}\right) \cdot \nu - d\left(1 - \frac{1}{r^{d+1}}\right) \\
&\geq 0,
\end{aligned}$$

where the first inequality holds since $\nu \leq 1$ and the second inequality is by Definition 4.2 and $\nu \geq \widehat{\nu}$. Since $x \geq 2$, we get

$$A(\nu) \cdot x^{t+1} + B(\nu) \cdot x^t \geq (2A(\nu) + B(\nu)) \cdot x^t.$$

To complete the proof we show that $2A(\nu) + B(\nu) \geq 0$ for $\nu \geq \widehat{\nu}$.

$$\begin{aligned}
&2A(\nu) + B(\nu) \\
&= \left(2^{d+1} + 2(d-1)\left(1 - \frac{1}{r^{t+1}}\right) - \frac{2}{r} - \frac{1}{2}(d+1)\left(1 - \frac{1}{r^t}\right) - \frac{2}{r}\left(\left(\frac{3}{2}\right)^t - 1\right)\right) \cdot \nu \\
&\qquad - 2d\left(1 - \frac{1}{r^{t+1}}\right) + \frac{1}{2}(d+1)\left(1 - \frac{1}{r^t}\right) \\
&= \left(2^{d+1} + 2(d-1)\left(1 - \frac{1}{r^{t+1}}\right) - \frac{1}{2}(d+1)\left(1 - \frac{1}{r^t}\right) - \frac{2}{r}\left(\frac{3}{2}\right)^t\right) \cdot \nu \\
&\qquad - 2d\left(1 - \frac{1}{r^{t+1}}\right) + \frac{1}{2}(d+1)\left(1 - \frac{1}{r^t}\right),
\end{aligned}$$

which again is a monotone increasing function in ν. Since $\widehat{\nu} \geq \frac{d}{2^d+d-1}$ by Lemma 4.3 and $\nu \geq \widehat{\nu}$, we get

$$\begin{aligned}
&2A(\nu) + B(\nu) \\
&\geq \left(2^{d+1} + 2(d-1)\left(1 - \frac{1}{r^{t+1}}\right) - \frac{1}{2}(d+1)\left(1 - \frac{1}{r^t}\right) - \frac{2}{r}\left(\frac{3}{2}\right)^t\right) \cdot \frac{d}{2^d + d - 1} \\
&\qquad - 2d\left(1 - \frac{1}{r^{t+1}}\right) + \frac{1}{2}(d+1)\left(1 - \frac{1}{r^t}\right) \\
&= \frac{(d+1)(2^d-1) \cdot r^{t+1} - 4d\left(\frac{3}{2}\right)^t \cdot r^t - (d+1)(2^d-1) \cdot r + 4d2^d}{2(2^d+d-1)r^{t+1}}.
\end{aligned}$$

Define $D(d,t)$ as the numerator of this term. Thus,

$$D(d,t) = (d+1)(2^d-1) \cdot r^{t+1} - 4d\left(\frac{3}{2}\right)^t \cdot r^t - (d+1)(2^d-1) \cdot r + 4d2^d.$$

If $r \le \frac{4}{3}$ then $\qquad D(d,t) \ge (d+1)(2^d-1)\cdot r(r^t-1) \ge 0,$

for all integer $d \ge 1$ and $0 \le t \le d$. If $r \ge \frac{4}{3}$ then

$$\begin{aligned} D(d,t) &\ge (d+1)(2^d-1)\cdot r(r^t-1) - 4d\left(\frac{3}{2}\right)^t \cdot (r^t-1) \\ &\ge \left((d+1)(2^d-1)\cdot\frac{4}{3} - 4d\left(\frac{3}{2}\right)^d\right)\cdot(r^t-1), \end{aligned}$$

which is nonnegative for all integer $d \ge 4$ and $0 \le t \le d$. For $t \le d \le 3$, $D(d,t) \ge 0$ can be checked using elementary calculus. ■

We are now ready to prove the upper bound of our main result.

Theorem 4.5 *For congestion games with polynomial cost functions with maximum degree d and nonnegative coefficients, we have*

$$PoS \le \max_{r>1} \frac{(2^d d + 2^d - 1)\cdot r^{d+1} - (d+1)\cdot r^d + 1}{(2^d + d - 1)\cdot r^{d+1} - (d+1)\cdot r^d + 2^d d - d + 1}.$$

Proof: Let s^* be an optimum assignment and let s be a pure Nash equilibrium with $\Phi(\mathsf{s}) \le \Phi(\mathsf{s}^*)$. Such a Nash equilibrium exists by (1). Define $x_e = n_e(\mathsf{s})$ and $y_e = n_e(\mathsf{s}^*)$. Then

$$\begin{aligned} SC(\mathsf{s}) &\le SC(\mathsf{s}) + (d+1)(\Phi(\mathsf{s}^*) - \Phi(\mathsf{s})) \\ &= \sum_{e\in E} n_e(\mathsf{s})\cdot c_e(n_e(\mathsf{s})) + (d+1)\sum_{e\in E}\left(\sum_{i=1}^{n_e(\mathsf{s}^*)} c_e(i) - \sum_{i=1}^{n_e(\mathsf{s})} c_e(i)\right) \\ &= \sum_{e\in E} x_e\cdot c_e(x_e) + (d+1)\sum_{e\in E}\left(\sum_{i=1}^{y_e} c_e(i) - \sum_{i=1}^{x_e} c_e(i)\right). \end{aligned} \tag{12}$$

Moreover, since s is a pure Nash equilibrium, we have

$$\begin{aligned} SC(\mathsf{s}) &= \sum_{i=1}^{n} C_i(\mathsf{s}) \le \sum_{i=1}^{n} C_i(s_i^*, \mathsf{s}_{-i}) \\ &\le \sum_{i=1}^{n}\sum_{e\in s_i^*} c_e(n_e(\mathsf{s})+1) = \sum_{e\in E} y_e\cdot c_e(x_e+1). \end{aligned} \tag{13}$$

Let $\widehat{\nu}$ as defined in Definition 4.2. Taking the convex combination $\widehat{\nu} \cdot (13) + (1-\widehat{\nu}) \cdot (12)$ of those inequalities gives

$$SC(\mathsf{s}) \leq \sum_{e \in E} \left[\widehat{\nu} \cdot y_e \cdot c_e(x_e+1) + (1-\widehat{\nu}) x_e \cdot c_e(x_e) + (1-\widehat{\nu})(d+1) \left(\sum_{i=1}^{y_e} c_e(i) - \sum_{i=1}^{x_e} c_e(i) \right) \right]$$

With $\lambda = d+1-d\widehat{\nu}$ and $\mu = (2^d+d-1)\widehat{\nu} - d$, applying Proposition 4.4 gives

$$SC(\mathsf{s}) \leq \sum_{e \in E} [\mu \cdot x_e \cdot c_e(x_e) + \lambda \cdot y_e \cdot c_e(y_e)] = \mu \cdot SC(\mathsf{s}) + \lambda SC(\mathsf{s}^*).$$

Thus,

$$\frac{SC(\mathsf{s})}{SC(\mathsf{s}^*)} \leq \frac{\lambda}{1-\mu} = \frac{d+1-d\widehat{\nu}}{d+1-(2^d+d-1)\widehat{\nu}},$$

By Definition 4.2, for all real numbers $r > 1$, we have

$$\widehat{\nu} \geq \frac{d(1-\frac{1}{r^{d+1}})}{2^d + (d-1)(1-\frac{1}{r^{d+1}}) - \frac{1}{r}}. \tag{14}$$

Denote $\hat{r}$ as the value for $r > 1$ which makes inequality (14) tight. Such a value $\hat{r}$ must exist since $\widehat{\nu}$ is the minimum value satisfying this inequality. So,

$$\widehat{\nu} = \frac{d(\hat{r}^{d+1}-1)}{2^d\hat{r}^{d+1} + (d-1)(\hat{r}^{d+1}-1) - \hat{r}^d}.$$

Substituting this in the bound from Theorem 4.5 gives

$$\begin{aligned} PoS &\leq \frac{d+1-d\widehat{\nu}}{d+1-(2^d+d-1)\widehat{\nu}} \\ &= \frac{(d+1)2^d\hat{r}^{d+1}+(d^2-1)(\hat{r}^{d+1}-1)-(d+1)\hat{r}^d-d^2(\hat{r}^{d+1}-1)}{2^d(d+1)\hat{r}^{d+1}+(d^2-1)(\hat{r}^{d+1}-1)-(d+1)\hat{r}^d-2^d d(\hat{r}^{d+1}-1)-d(d-1)(\hat{r}^{d+1}-1)} \\ &= \frac{(2^d d + 2^d - 1)\hat{r}^{d+1} - (d+1)\hat{r}^d + 1}{(2^d+d-1)\hat{r}^{d+1} - (d+1)\hat{r}^d + 2^d d - d + 1} \\ &\leq \max_{r>1} \frac{(2^d d + 2^d - 1) \cdot r^{d+1} - (d+1) \cdot r^d + 1}{(2^d+d-1) \cdot r^{d+1} - (d+1) \cdot r^d + 2^d d - d + 1}, \end{aligned}$$

which proves the upper bound. ■

5 Separation

For the linear case, the Price of Stability was equal to the Price of Anarchy of dominant strategies, as the matching lower bound instance would hold for dominant strategies. Here, we show that linear functions was a degenerate case, and that this is not true for higher order polynomials. We show that for games that possess dominant equilibria, the Price of Anarchy for them is strictly smaller[3]. Our separation leaves as an open question what is the exact value of the Price of Anarchy of dominant strategies for these games. Our separation result uses the following technical proposition, the proof of which can be found in the appendix.

Proposition 5.1 Let $\lambda = \frac{7}{4}, \mu = \frac{1}{4}$ and $\nu = \frac{3}{4}$. Then for all quadratic cost functions c with nonnegative coefficients and for all nonnegative integers x, y we have

$$\nu \cdot y \cdot c(y) + y \cdot c(x+1) - \nu \cdot x \cdot c(y+1) \leq \mu \cdot x \cdot c(x) + \lambda \cdot y \cdot c(y).$$

Theorem 5.2 *Consider a congestion game with quadratic cost functions which admits a dominant strategy equilibrium* s. *Then* $\frac{SC(\mathsf{s})}{\text{OPT}} \leq \frac{7}{3}$.

Proof: Let s^* be an optimum assignment and let s be the dominant strategy equilibrium. Define $x_e = n_e(\mathsf{s})$, $y_e = n_e(\mathsf{s}^*)$ and $\Delta_e(x) = c_e(x+1) - c_e(x)$. Since s is a dominant equilibrium we have $C_i(\mathsf{s}^*_{-i}, s_i) \leq C_i(\mathsf{s}^*)$ for every player i. Thus,

$$\begin{aligned} SC(\mathsf{s}^*) = \sum_{i=1}^{n} C_i(\mathsf{s}^*) &\geq \sum_{i=1}^{n} C_i(\mathsf{s}^*_{-i}, s_i) \\ &= \sum_{i=1}^{n} \sum_{e \in s_i \cap s_i^*} c_e\left(n_e(\mathsf{s}^*)\right) + \sum_{i=1}^{n} \sum_{e \in s_i \setminus s_i^*} c_e\left(n_e(\mathsf{s}^*) + 1\right) \\ &= \sum_{e \in E} n_e(\mathsf{s}) c_e\left(n_e(\mathsf{s}^*) + 1\right) - \sum_{i=1}^{n} \sum_{e \in s_i \cap s_i^*} \Delta_e\left(n_e(\mathsf{s}^*)\right) \\ &= \sum_{e \in E} x_e c_e\left(y_e + 1\right) - \sum_{i=1}^{n} \sum_{e \in s_i \cap s_i^*} \Delta_e\left(y_e\right). \end{aligned} \tag{15}$$

[3]By a more elaborate analysis one can come up with an upper bound of ≈ 2.242 Here we just wanted to demonstrate the separation of the two measures.

Following similar steps, since s is a pure Nash equilibrium, we have

$$SC(\mathsf{s}) = \sum_{i=1}^{n} C_i(\mathsf{s}) \leq \sum_{i=1}^{n} C_i(\mathsf{s}_{-i}, s_i^*) = \sum_{e \in E} y_e c_e \left(x_e + 1\right) - \sum_{i=1}^{n} \sum_{e \in s_i \cap s_i^*} \Delta_e \left(x_e\right). \tag{16}$$

Summing up (15) and (16), we end up with

$$\begin{aligned}
&SC(\mathsf{s}) \\
&\leq SC(\mathsf{s}^*) + \sum_{e \in E} \left(y_e c_e \left(x_e + 1\right) - x_e c_e \left(y_e + 1\right)\right) + \sum_{i=1}^{n} \sum_{e \in s_i \cap s_i^*} \left(\Delta_e \left(y_e\right) - \Delta_e \left(x_e\right)\right) \\
&\leq SC(\mathsf{s}^*) + \sum_{e \in E} \left(y_e c_e \left(x_e + 1\right) - x_e c_e \left(y_e + 1\right)\right). \qquad (17)
\end{aligned}$$

Taking the convex combination $\nu \cdot (17) + (1 - \nu) \cdot (16)$ gives

$$\begin{aligned}
&SC(\mathsf{s}) \\
&\leq \nu \cdot \left(SC(\mathsf{s}^*) + \textstyle\sum_{e \in E} \left(y_e c_e \left(x_e + 1\right) - x_e c_e \left(y_e + 1\right)\right)\right) + (1 - \nu) \cdot \left(\textstyle\sum_{e \in E} y_e c_e \left(x_e + 1\right)\right) \\
&= \nu \cdot SC(\mathsf{s}^*) + \sum_{e \in E} \left(y_e c_e \left(x_e + 1\right) - \nu \cdot \left(x_e c_e \left(y_e + 1\right)\right)\right). \qquad (18)
\end{aligned}$$

With $\lambda = 7/4$, $\mu = 1/4$, and $\nu = 3/4$, applying Proposition 5.1 gives

$$\begin{aligned}
SC(\mathsf{s}) &\leq \sum_{e \in E} \left[\mu \cdot x_e \cdot c_e(x_e) + \lambda \cdot y_e \cdot c_e(y_e)\right] \\
&= \mu \cdot SC(\mathsf{s}) + \lambda SC(\mathsf{s}^*).
\end{aligned}$$

Thus,

$$\frac{SC(\mathsf{s})}{SC(\mathsf{s}^*)} \leq \frac{\lambda}{1 - \mu} = \frac{7}{3} < 2.36,$$

which proves the theorem. ■

Observe that this upper bound is strictly smaller than the exact value of the PoS for *general* congestion games with quadratic cost functions from Theorem 1.1, which was ≈ 2.36.

References

[1] S. Aland, D. Dumrauf, M. Gairing, B. Monien, and F. Schoppmann. Exact price of anarchy for polynomial congestion games. *SIAM Journal on Computing*, 40(5):1211–1233, 2011.

[2] Susanne Albers. On the value of coordination in network design. In *SODA '08: Proceedings of the nineteenth annual ACM-SIAM symposium on Discrete algorithms*, pages 294–303, Philadelphia, PA, USA, 2008. Society for Industrial and Applied Mathematics.

[3] Elliot Anshelevich, Anirban Dasgupta, Jon M. Kleinberg, Éva Tardos, Tom Wexler, and Tim Roughgarden. The price of stability for network design with fair cost allocation. In *45th Symposium on Foundations of Computer Science (FOCS)*, pages 295–304, 2004.

[4] Baruch Awerbuch, Yossi Azar, and Amir Epstein. Large the price of routing unsplittable flow. In *Proceedings of the 37th Annual ACM Symposium on Theory of Computing (STOC)*, pages 57–66, 2005.

[5] Kshipra Bhawalkar, Martin Gairing, and Tim Roughgarden. Weighted congestion games: Price of anarchy, universal worst-case examples, and tightness. In *Algorithms - ESA 2010, 18th Annual European Symposium*, pages 17–28, 2010.

[6] V. Bilò. A Unifying Tool for Bounding the Quality of Non-Cooperative solutions in Weighted Congestion Games. In *Proceedings of the 10th International Workshop on Approximation and Online Algorithms (WAOA'12)*, pages 215–228, 2012.

[7] Vittorio Bilò and Roberta Bove. Bounds on the price of stability of undirected network design games with three players. *Journal of Interconnection Networks*, 12(1-2):1–17, 2011.

[8] Vittorio Bilò, Ioannis Caragiannis, Angelo Fanelli, and Gianpiero Monaco. Improved lower bounds on the price of stability of undirected network design games. In *Algorithmic Game Theory - Third International Symposium (SAGT)*, pages 90–101, 2010.

[9] I. Caragiannis, M. Flammini, C. Kaklamanis, P. Kanellopoulos, and L. Moscardelli. Tight Bounds for Selfish and Greedy Load Balancing. *Algorithmica*, 61(3):606–637, 2011.

[10] Ho-Lin Chen and Tim Roughgarden. Network design with weighted players. In *SPAA '06: Proceedings of the eighteenth annual ACM symposium on Parallelism in algorithms and architectures*, pages 29–38, New York, NY, USA, 2006. ACM.

[11] G. Christodoulou and E. Koutsoupias. The Price of Anarchy of Finite Congestion Games. In *Proceedings of the 37th Annual ACM Symposium on Theory of Computing (STOC'05)*, pages 67–73, 2005.

[12] G. Christodoulou, E. Koutsoupias, and P.G. Spirakis. On the performance of approximate equilibria in congestion games. *Algorithmica*, 61(1):116–140, 2011.

[13] George Christodoulou, Christine Chung, Katrina Ligett, Evangelia Pyrga, and Rob van Stee. On the price of stability for undirected network design. In *7th Workshop on Approximation and Online Algorithms (WAOA)*, 2009.

[14] George Christodoulou and Elias Koutsoupias. On the price of anarchy and stability of correlated equilibria of linear congestion games. In *Algorithms - ESA 2005, 13th Annual European Symposium*, pages 59–70, 2005.

[15] Yann Disser, Andreas Emil Feldmann, Max Klimm, and Matús Mihalák. Improving the h_k-bound on the price of stability in undirected shapley network design games. *CoRR*, abs/1211.2090, 2012.

[16] Amos Fiat, Haim Kaplan, Meital Levy, Svetlana Olonetsky, and Ronen Shabo. On the price of stability for designing undirected networks with fair cost allocations. In *ICALP (1)*, pages 608–618, 2006.

[17] E. Koutsoupias and C. H. Papadimitriou. Worst-case equilibria. In *Proceedings of the 16th Annual Symposium on Theoretical Aspects of Computer Science (STACS)*, pages 404–413, 1999.

[18] Euiwoong Lee and Katrina Ligett. Improved bounds on the price of stability in network cost sharing games. In *ACM Conference on Electronic Commerce*, pages 607–620, 2013.

[19] J. Li. An O (lognloglogn) upper bound on the price of stability for undirected Shapley network design games. *Information Processing Letters*, 2009.

[20] D. Monderer and L. Shapley. Potential games. *Games and Economics Behavior*, 14:124–143, 1996.

[21] R. W. Rosenthal. A class of games possessing pure-strategy Nash equilibria. *International Journal of Game Theory*, 2:65–67, 1973.

[22] T. Roughgarden. Intrinsic Robustness of the Price of Anarchy. *Communications of the ACM*, 55(7):116–123, 2012.

[23] A. S. Schulz and N. E. Stier Moses. On The Performance of User Equilibria in Traffic Networks. In *Proceedings of the 14th Annual ACM-SIAM Symposium on Discrete Algorithms (SODA'03)*, pages 86–87, 2002.

Appendix

6 Material for proof of Proposition 4.4

The following is used in the proof of Proposition 4.4.

Lemma 6.1 *Define $\lambda = d + 1 - d\nu$ and $\mu = (2^d + d - 1)\nu - d$ and fix a maximum degree d. Then for all nonnegative integers $x \leq y$ and for all $0 \leq t \leq d$, we have*

$$(\mu + \nu - 1) \cdot x^{t+1} + \lambda \cdot y^{t+1} - \nu \cdot y(x+1)^t + (1-\nu)(d+1)\left(\sum_{i=1}^{x} i^t - \sum_{i=1}^{y} i^t\right) \geq 0. \quad (19)$$

Proof: We consider the cases $x < y$ and $x = y$ separately.
Case $x < y$:
In this case

$$y(x+1)^t \leq y^{t+1} \quad (20)$$

and

$$\sum_{i=1}^{x} i^t - \sum_{i=1}^{y} i^t = - \sum_{i=x+1}^{y} i^t \geq (x-y)y^t \geq x^{t+1} - y^{t+1}. \quad (21)$$

Using (20),(21) and the above values for λ, μ, we can lower bound the left-hand-side of (19) by

$$\begin{aligned}
&(\mu + \nu - 1) \cdot x^{t+1} + \lambda \cdot y^{t+1} - \nu \cdot y^{t+1} + (1-\nu)(d+1)\left(x^{t+1} - y^{t+1}\right) \\
&= (\mu + \nu - 1 + d + 1 - \nu d - \nu) \cdot x^{t+1} + (\lambda - \nu - d - 1 + \nu d + \nu) \cdot y^{t+1} \\
&= (\mu + d - \nu d) \cdot x^{t+1} + (\lambda - d - 1 + \nu d) \cdot y^{t+1} \\
&= (2^d - 1)\nu \cdot x^{t+1} \\
&\geq 0,
\end{aligned}$$

as needed.
Case $x = y$:
Using the binomial theorem we can bound

$$(x+1)^t = \sum_{i=0}^{t} \binom{t}{i} \cdot x^{t-i} \leq x^t \cdot \sum_{i=0}^{t} \binom{t}{i} = 2^t \cdot x^t \leq 2^d \cdot x^t.$$

With the assumption $x = y$, this implies

$$\begin{aligned}
&(\mu+\nu-1)\cdot x^{t+1} + \lambda\cdot y^{t+1} - \nu\cdot y(x+1)^t \\
&= (\mu+\nu-1+\lambda)\cdot x^{t+1} - \nu\cdot x(x+1)^t \\
&\geq (\mu+\nu-1+\lambda-2^d\nu)\cdot x^{t+1} \\
&= 0,
\end{aligned}$$

as needed. ■

7 Proof of Proposition 5.1

Proof: It is sufficient to prove the inequality for monomials of degree 0, 1, and 2 respectively. For $d = 0$, this is trivially true. For $d = 1$, this is equivalent to

$$\frac{1}{4}\cdot x^2 + y^2 + \frac{3}{4}\cdot x(y+1) - y(x+1) \geq 0,$$

and this is true, since

$$x^2 + 4y^2 - 4y(x+1) + 3x(y+1) = (x-2y)^2 + y(3x-4) + 3x \geq 0.$$

The last inequality is easy to check, since for $x \geq 2$ all terms are nonnegative, and the cases $x = 0, x = 1$ are trivially true.

For $d = 2$, we need to show that

$$f(x,y) = x^3 + 4y^3 - 4y(x+1)^2 + 3x(y+1)^2 \geq 0.$$

It is trivial to verify the inequality for $y = 0$ and is easy to see that $f(x,1) = x(x-2)^2 \geq 0$. Hence, it suffices to show the inequality for $y \geq 2$.

We can rewrite $f(x,y)$ in the following way

$$f(x,y) = x\left(x - \frac{5}{2}y\right)^2 + A(y),$$

where

$$A(y) := yx^2 + \left(-\frac{13}{4}\,y^2 + 3 - 2\,y\right)x + 4\,y^3 - 4\,y.$$

It suffices to show $A(y) \geq 0$ for $y \geq 2$. However, for $y = 2$, it is $A(2) = 2(x-3)(x-4) \geq 0$.

Therefore, it is enough to show $A(y) \geq 0$ for $y \geq 3$. This is a quadratic function with respect to x, with discriminant

$$\Delta = -\left(\frac{87}{16}\,y^4 - 13\,y^3 - 1/2\,y^2 + 12\,y - 9\right).$$

This can be easily shown to be always negative for $y \geq 3$ (for example by iteratively substituting $y^d \geq 3y^{d-1}$ for $d = 4$ down to $d = 1$, as the coefficient of d always remains positive in every step). ■

We are now ready to prove Theorem 5.2.